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Differential Calculus

Interval

1. Open - it is represented as (a,b).
   - $(a,b) = \{x \mid a < x < b\}$

2. Closed - its represented as $[a,b]$.
   - $[a,b] = \{x \mid a \leq x \leq b\}$

Continuous function

A function said to be $y = f(x)$ is said to be continuous at $x = a$. If the following continuous and satisfies

a) $f(a)$ exists

b) $\lim_{x \to a^-}$ and $\lim_{x \to a^+} f(x)$ exists and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

c) $\lim_{x \to a} f(x) = f(a)$

REMARKS:

1. Algebraic form of polynomial, trigonometric, inverse trig., exponentials, logs are always continuous

2. Addition, subtraction, product and ratio of 2 or more continuous function are also continuous.

Differential function

A func^n $y = f(x)$ is said to be differentiable at $x = a$ if $\lim_{x \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

• always continuous but all continuous are not differentiable. (eg.:?)

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• not continuous implies not differentiable.

Formulae to remember

1. $\frac{d(x^n)}{dx} = n x^{n-1}$

2. $\frac{d(e^{ax})}{dx} = a e^{ax}$

3. $\frac{d(a^x)}{dx} = a^x \log a$

4. $\frac{d(\log x)}{dx} = \frac{1}{x}$

5. $\frac{d(\sin x)}{dx} = \cos x$

6. $\frac{d(\cos x)}{dx} = -\sin x$

7. $\frac{d(\tan x)}{dx} = \sec^2 x$

8. $\frac{d(\cot x)}{dx} = -\csc^2 x$

9. $\frac{d(\sec x)}{dx} = \sec x \tan x$

10. $\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

11. $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

12. $\frac{d(\csc x)}{dx} = -\cot x \csc x$

13. $\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$

14. $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$

Rolle's Mean Value Theorem

If function $f(x)$ is

- Step I

a. continuous is closed interval $[a,b]$

b. differentiate in open interval (a,b)

c. $f(a) = f(b)$

- Step II

a. there exists at least one point

b. $x = c$ in (a,b) such that $f'c = 0$

Example 1) Verify Rolle's Mean Value Theorem for the function $f(x) = x^2$ in $[-1, 1]$

A Step I:

a. Since x^2 is polynomial function is continuous in $[-1, 1]$

$\therefore f(x) = x^2$ is continuous in $[-1, 1]$

b. $f(x) = x^2$

$f'(x) = 2x$ is finite in $(-1, 1)$

$\therefore f(x)$ is a differentiable function in $(-1, 1)$

c. $f(-1) = (-1)^2 = 1$

$f(1) = (1)^2 = 1$

$f(-1) = f(1)$

Thus $f(x) = x^2$ satisfies all the three conditions for Rolle's Theorem.

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Step II :

a. $\therefore$ there exists at least one point $x = c$ in $(-1, 1)$ such that $f'(c) = 0$

$\therefore f(x) = x^2$

$f(c) = c^2$

$f'(c) = 2c$

But $f'(c) = 0$

$\Rightarrow 2c = 0$

$\Rightarrow c = 0$

$c = 0 \in (-1, 1)$

$\therefore$ Rolle's Theorem is verified.

Hence Proved.

Example 2) Verify Rolle's Mean Value Theorem for the function $f(x) = x - x^2$ in $[0, 1]$.

Step I :

a. Since $x - x^2$ is polynomial in continuous in $[0, 1]$

$\therefore f(x) = x - x^2$ is continuous in $[0, 1]$.

b. $f(x) = x - x^2$

$f'(x) = 1 - 2x$ is finite in $(0, 1)$

$\therefore f(x)$ is differentiable function in $(0, 1)$

c. $f(0) = 0 - 0^2 = 0$

$f(1) = 1 - 1^2 = 0$

$f(0) = f(1)$

$\therefore f(x) = x - x^2$ satisfies all the three condition of Rolle's theorem.

Step II :

a. $\therefore$ there exists at least one point $x = c$ in $(0,1)$ such that $f'(c) = 0$

$f(x) = x - x^2$

$f(c) = c - c^2$

$f'(c) = 1 - 2c$

But $f'(c) = 0$

$1 - 2c = 0$

$1 = 2c$

$c = \frac{1}{2} = 0.5$

$\therefore c = 0.5 \in [0,1]$

Rolle's theorem is verified

Q. Verify ~~Rolle's theorem~~ RMV 'T' $f(x) = x(x-1)e^{-x}$ in $[0,1]$.

Ans. ~~$\because x(x-1)e^{-x}$ is a polynomial is continuous in $[0,1]$~~

Step 1:

a. $[x(x-1)e^{-x}]$ is a polynomial is continuous ~~at~~ in $[0,1]$.

$\therefore f(x) = x(x-1)e^{-x}$ is continuous at $[0,1]$

b. $f(x) = x(x-1)e^{-x} = (x^2 - x)e^{-x}$

$f'(x) = +(2x-1)e^{-x} - (x^2 - x)e^{-x}$ is finite in $(0,1)$

$\therefore f(x)$ is differentiable function in $(0,1)$

c. $f(x) = x(x-1)e^{-x}$

$f(0) = (0)(0-1)(e)^{-0} = 0$

$f(1) = (1)(1-1)(e)^{-1} = 0$

$f(0) = f(1)$

$\therefore f(x) = x(x+1)e^{-x}$ satisfies all the conditions of RMV 'T'.

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Step II:

a. $\therefore$ there exists at least one point $x = c$ in $(0,1)$ such that $f'(c) = 0$.

$$f(x) = x(x-1)e^{-x} = (x^2-x)e^{-x}$$
$$f(c) = \cancel{c(c-1)}\ (c^2-c)e^{-c}$$
$$f'(c) = (2c-1)e^{-c} - (c^2-c)e^{-c}$$

But $f'(c) = 0$

$$\cancel{(2c-1)e^{-c} - (c^2-c)e^{-c} = 0}$$
$$\cancel{2ce^{-c} - e^{-c} - c^2e^{-c} + ce^{-c}} = 0$$
$$\cancel{2ce^{-c} = e^{-c}}$$
$$\cancel{c = 0.5}$$

$$(2c-1)e^{-c} - (c^2-c)e^{-c} = 0$$
$$[(2c-1) - (c^2-c)]e^{-c} = 0$$
$$[-2c + 1 + c^2 + c]e^{-c} = 0$$
$$[c^2 - 3c + 1]e^{-c} = 0$$
$$c^2 - 3c + 1 = e^{-c}$$

$$\therefore c = 2.61$$
$$c = 0.38$$

b. $c = 2.61 \notin (0,1)$

$c = 0.38 \in (0,1)$

Hence Proved.

Q. $f(x) = e^{-x} \sin x$ in $[0, \pi]$.

Ans. Step I:

a. $\because e^{-x}$ ~~sin x~~ is ~~a polynomial~~ exponential. and $\sin x$ is trigonometric.

$\therefore$ Function is continuous in $[0, \pi]$

function is continuous in $[0, \pi]$ ~~and since~~

b. $f(x) = e^{-x} \sin x$

$f'(x) = e^{-x} \cos x - e^{-x} \sin x$ finite in $(0,\pi)$

$f(x)$ is a differentiable func^n in $(0,\pi)$

c. $f(0) = e^{-0} \sin(0) = 0$

$f(\pi) = e^{-\pi} \sin(\pi) = 0$

$f(0) = f(\pi)$

$\therefore f(x) = e^{-x} \sin x$ satisfies all the three condi-tions

Step II:

a. $\therefore$ there exists at least one point $x = c$ in $(0,\pi)$ such that $f'(c) = 0$

$f(x) = e^{-x} \sin x$

$f(c) = e^{-c} \sin c$

$f'(c) = -e^{-c} \sin c - e^{-c} \cos c$

But $f'(c) = 0$

$e^{-c} \sin c - e^{-c} \cos c = 0$

$e^{-c} (\sin c - \cos c) = 0$

$\sin c = \cos c$

$c = \pi/4 = 45^\circ \in (0,\pi)$

Hence Proved.

Q. Let $f(x) = \sin x + \cos x - 1$ in $[0, \pi/2]$

Ans. Step I:

a. $\because -1$ is algebraic function is continuous in $[0, \pi/2]$ and $\sin x$ and $\cos x$ are trig. func^n is continuous in $[0, \pi/2]$

$\therefore f(x) = \sin x + \cos x - 1$

$\therefore f'(x) = \cos x - \sin x$ is finite in $(0, \pi/2)$

$\therefore f(x)$ is differentiable function in $(0, \pi/2)$

$f(0) = \cos 0 - \sin 0 - 1 = 0$

$f(\pi/2) = \cos \pi/2 - \sin \pi/2 - 1 = 0$

$f(0) = f(\pi/2)$

Thus, $f(x) = \sin x + \cos x - 1$ satisfies all the three conditions of Rolle's theorem

Step II:

$\therefore$ there exists at least one point $x = c$ in $(0, \pi/2)$ such that $f'(c) = 0$

$\therefore f(x) = \sin x + \cos x - 1$

$f(c) = \sin c + \cos c - 1$

$f'(c) = \cos c - \sin c + 0$

But $f'(c) = 0$

$\cos c - \sin c = 0$

$\cos c = \sin c$

$\Rightarrow c = \frac{\pi}{4} = 45 \in (0, \pi/2)$

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$$\lim_{x\to 0} \frac{a\cos x - a + bx^2}{x^4} = \frac{1}{12}.$$

Ans. Let $L = \lim_{x\to 0} \frac{a\cos x - a + bx^2}{x^4}$ where $L = \frac{1}{12}$.

Apply L'Hospital Rule,

$$L = \lim_{x\to 0} \frac{-a\sin x + 2bx}{4x^3} = \frac{0}{0}$$

Apply L'Hospital Rule,

$$L = \lim_{x\to 0} \frac{-a\cos x + 2b}{12x^2}$$

which is infinite but $L = 1/12$ [given]

$\therefore$ This is possible when $-a + 2b = 0$

$$2b = a \quad \text{—— (1)}$$

Put $a = 2b$ in eq (1),

$$L = \lim_{x\to 0} \frac{-2b\cos x + 2b}{12x^2}$$

Apply L'Hospital Rule,

$$L = \lim_{x\to 0} \frac{+2b\sin x}{24x}$$

$$= \frac{b}{12} \lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$$

$$L = \frac{b}{12}$$

$$\frac{1}{12} = \frac{b}{12}$$

$$\boxed{\therefore b = 1}$$

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Indeterminant form (0^0, ∞^0, 1^∞)

Let $L = \lim_{x \to a} [f(x)]^{g(x)}$

Taking log on both sides,

$$\log L = \lim_{x \to a} [g(x) \cdot \log (f(x))]$$

$$= \lim_{x \to a} \left[\frac{\log f(x)}{\frac{1}{g(x)}} \right]$$

$\log L = a$ (say)

$L = e^a$

Q. $\lim_{x \to \pi/2} (\cos x)^{\cos x}$

Taking log on both sides,

$$\log L = \lim_{x \to \pi/2} [\cos x \cdot \log (\cos x)]$$

$$= \lim_{x \to \pi/2} \left[\frac{\log (\cos x)}{\frac{1}{\cos x}} \right]$$

$$= \lim_{x \to \pi/2} \left[\frac{\log (\cos x)}{\sec x} \right]$$

$$= \lim_{x \to \pi/2} \left[\frac{\frac{1}{\cos x} (-\sin x)}{\sec x \cdot \tan x} \right]$$

$$= \lim_{x \to \pi/2} \left[\frac{-\tan x}{\tan x \cdot \sec x} \right]$$

$$= \lim_{x \to \pi/2} (-\cos x)$$

$\log L = 0$

$L = e^0$

$L = 1$

Q. Let $L = \lim_{x \to 0} (\cot x)^{\sin x}$

Taking log on both sides,

$$\log L = \lim_{x \to 0} (\sin x \log(\cot x))$$

$$= \lim_{x \to 0} \left[\frac{\log(\cot x)}{\frac{1}{\sin x}} \right]$$

$$= \lim_{x \to 0} \left[\frac{\log(\cot x)}{\csc x} \right]$$

Apply L'Hospital Rule

$$L = \lim_{x \to 0} \left[\frac{\frac{1}{\cot}(-\csc^2 x)}{-\csc x \cdot \cot x} \right]$$

$$= \lim_{x \to 0} (\tan x \cdot \csc x \cdot \tan x)$$

$$= \lim_{x \to 0} \left(\tan \cdot \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to 0} (\tan x \cdot \sec x)$$

$\log L = 0$

$L = e^0$

$L = 1$

Q. Let $L = \lim_{x \to 0} (\log x)^{\frac{1}{1-\log x}}$

Taking log on both sides,

$$\log L = \lim_{x \to e} \left[\frac{1}{1 - \log x} \right]$$

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Type II : $-L \le x \le L$

Even and odd

The fourier's series, of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

1) $f(x)$ is even function

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = 0$$

2) $f(x)$ is odd junction

$$a_0 = 0$$

SharkCoders

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx$$

$f(x) = x^2 \qquad -1 \le x \le 1$

$L = 1$

$\therefore f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

$f(x) = f(-x)$ is even.

$\therefore B_n = 0$

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$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 x^2\,dx$$

$$= \cancel{2} \; 2\left[\frac{x^3}{3}\right]_0^1$$

$$a_0 = 2\left[\frac{1}{3} - 0\right] = \frac{2}{3}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$= \frac{2}{1}\int_0^L x^2\cdot\cos n\pi x\;dx$$

$u = x^2$

$v = \cos(n\pi x)$

$$a_n = 2\left[x^2\left(\sin\left(\frac{n\pi x}{n\pi}\right)\right) - 2x\left(-\cos\left(\frac{n\pi x}{n^2\pi^2}\right)\right) + 2\left(-\sin\left(\frac{n\pi x}{L}\right)\right)\right]_0^1$$

$$a_n = 2\left[\frac{\sin(n\pi)}{n\pi} + 2\frac{\cos n\pi}{n^2\pi^2} - 2\frac{\sin n\pi}{n^3\pi^3}\right]$$

$$a_n = 2\left[\frac{2(-1)^n}{n^2\pi^2}\right]$$

$$a_n = \frac{4(-1)^n}{n^2\pi^2}$$

Q. $f(x) = x^2 \qquad -\pi \leq x \leq \pi$

Here $L = \pi$

$f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

$f(x)$ is even function

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cdot\cos nx \, dx$$

$u = x^2$

$v = \cos(nx)$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2\frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3} - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

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$$a_n = \frac{4(-1)^2}{n^2} = \frac{4}{n^2}$$

Q. $f(x) = x \qquad -1 \le x \le 1$

Here $L = 1$

$$f(x) = x$$
$$f(-x) = -x$$
$$= -x$$
$$= -f(x)$$

$$a_0 = 0$$
$$a_n = 0$$

$$b_n = \frac{2}{L}\int f(x)\cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= 2\int_0^1 x \cdot \sin(n\pi x)\, dx$$

$$= 2\left[x\cdot\left(\frac{-\cos n\pi x}{n\pi}\right) - 1\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[\left(\frac{-\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2}\right)\right]$$

$$= 2\left[\frac{-(-1)^2}{n\pi}\right]$$

$$b_n = \frac{-2(-1)^n}{n\pi}$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n}\right]$$

$$b_n = \frac{2}{n}(-1)^{n+1}$$

Half Range Fourier's Series

- Half length range cosine func^n

→ if $f(x)$ is defined in $0 \le x \le L$, then half range cosine series is given by ~~$f(x) = a_0$~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cdot \text{ where}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)\cdot dx \qquad -L < x < L$$

- Half range sine func^n

→ if $f(x)$ is defined in $0 \le x \le L$, then half range sine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot dx$$

Q. $f(x) = x^2$

half range Fourier cosine series $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{2}{L}\int_0^L f(x)\, dx$

$$a_n = \frac{2}{L}\int_0^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

Here $L = 1$

$$a_0 = 2\int_0^1 x^2\, dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{2}{1}\int_0^1 x^2 \cos n\pi x\, dx$$

$\because u = x^2$

$v = \cos n\pi x$

$$a_n = 2\left[x^2\left(\frac{\sin(n\pi x)}{n\pi}\right) - 2x\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\frac{\sin n\pi}{n\pi} + \frac{2\cos n\pi}{n^2\pi^2} - \frac{2\sin n\pi}{n^3\pi^3}\right]$$

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$$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Here $L = \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \cdot dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\pi^3}{3}\right)$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx$$

$\because$ $u = x^2$

$v = \cos nx$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin(n\pi)}{n\pi} + \frac{2\pi \cos n\pi}{n^2\pi^2} - 2\frac{\sin n\pi}{n^3\pi^3}\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2}\right]$$

$$a_n = \frac{4(-1)^n}{n^2}$$

Fourier series is

$$f(x) = \frac{1}{2} \times \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Q. Find HRF sign series for $f(x) = x^2$ in $0 < x < \pi$

Ans. $f(x) = x^2$ in $0 < x < \pi$

The HRF sign series $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

Here $L = \pi$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin\left(\frac{\cancel{\pi} n x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin(nx)\, dx$$

$$= \frac{2}{\pi} \left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\frac{-\pi^2 \cos n\pi}{n} + \frac{2\pi \sin n\pi}{n^2} + \frac{2 \cos n\pi}{n^3} \right) - \left(\frac{2}{n^3}\right) \right]$$

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$$= \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\pi^2(-1)^{n+1}}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]\sin(nx)$$

Q. Find HRF sign series $f(x) = x^2$ in $0 < x < 1$.

Q. Find HRF cosine ~~sign~~ series $f(x) = x$ in $0 < x < 1$

Q. Find HRF ~~sin~~sign series $f(x) = x$ in $0 < x < \pi$

Q. Find HRF ~~sing~~ sign series $f(x) = x$ in $0 < x < 1$

Q. Find HRF sign series $f(x) = x$ in $0 < x < \pi$

Q. Find HRF cosine series $f(x) = 1 - x$ in $0 < x < 1$

Ans. ~~$f(x) = \frac{2}{L}\int_0^L$~~

Let $f(x) = 1 - x$ in $0 < x < 1$

The HRF cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$$a_n = \frac{2}{L}\int_0^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

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Here $L = 1$

$$a_0 = 2\int_0^1 (1-x)\,dx$$

$$= 2\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[1 - \frac{1}{2}\right]$$

$$= 2 - 1 = 1$$

$$a_0 = 1$$

$$a_n = \frac{2}{1}\int_0^1 (1-x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= 2\int_0^1 (1-x)\cos(n\pi x)\,dx$$

$\because\ u = 1-x$

$v = \cos(n\pi x)$

$$a_n = 2\left[(1-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[\left(0 - \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 - \frac{1}{n^2\pi^2}\right)\right]$$

$$= 2\left[\frac{-(-1)^n}{n^2\pi^2} + \frac{1}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi^2}\left[1 - (-1)^n\right] =$$

Fourier series is:

$$f(x) = \cancel{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} (1-(-1)^n) \cos(n\pi x$$

$$1-x = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2}\right] \cos n\pi x$$

Q. Find the HRF sine series $f(x) = 1-x \quad 0 < x < 1$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)\cdot dx.$$

Here $L = 1$

$$\cancel{a_0 = 2\int_0^1 (1-x)\cdot\cos\left(\frac{n\pi x}{1}\right) d}$$

$$a_0 = \frac{2}{1}\int_0^1 (1-x)\,dx$$

$$= 2\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\frac{1}{2}\right] \frac{1}{2} = 1$$

$$a_0 = 1$$

$$a_n = \frac{2}{L}\int_0^1 f(x)\cdot\cos(n\pi x)\cdot dx$$

$$= 2\int_0^1 (1-x)\cos(n\pi x)\,dx$$

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$$= 2 \int \left[(1- \right.$$

$$a_n = 2 \int_0^1 (1-x) \cdot \cos(n\pi x)\, dx$$

$$= 2 \left\{ \cancel{\left[(1-x)\right]\left[\sin(n\pi x) \cdot n\pi x^2\right]} \right.$$

Fourier (Full / half range)

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Harmonic Analysis

[Thi]s Fourier series expansion of $f(x)$ in its range [giv]en by

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $a_0 = 2 \times [\text{mean value of } y = f(x)]$

$$a_n = 2 \times \left[\text{mean value of } y \cdot \cos\frac{n\pi x}{L}\right]$$

$$b_n = 2 \times \left[\text{mean value of } y \cdot \sin\left(\frac{n\pi x}{L}\right)\right]$$

If term $\left\{a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right\}$ is called n^{th} harmonic.

Half Range Cosine Series upto n-harmonic

- HRCS of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Half Range Sine Series upto n-harmonic

- HRSS of $f(x)$ is given by

$$f(x) = b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = 2 \times \left[\text{Mean value of } y \sin\left(\frac{n\pi x}{L}\right)\right]$$

$a_0 = 2 \times$ [mean value of $y = f(x)$]

$= 2 \times \frac{\Sigma y}{N}$ [N = no. of observation]

Example Obtain the constant term and the coefficient of first sine and cosine term in the fourier expansion of y as given in the following table

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Ans.

NOTE: Periodic funcⁿ means after a certain period the value of funcⁿ repeats

Range $\equiv (0, 6)$

$\therefore \; 2L = 6$ (period)

$L = 3$ — (1)

The Fourier series expansion of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

From (1),

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right)\right)$$

~~$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\right.$~~

$$a_0 = 2 \times \frac{\Sigma y}{N} = 2 \times \frac{\Sigma y}{6} = \frac{\Sigma y}{3} \quad \text{— (2)}$$

$$a_1 = 2 \times \left[\text{mean value of } y \cdot \cos\frac{\pi x}{3}\right]$$

$$= 2 \times \frac{\Sigma y \cdot \cos\frac{\pi x}{3}}{6} = \frac{\Sigma y \cdot \cos\frac{\pi x}{3}}{3} \quad \text{—— (3)}$$

$$b_1 = 2 \times \left[\text{mean value of } y \cdot \sin\frac{\pi x}{3}\right]$$

$$= 2 \times \frac{\Sigma y \sin\frac{\pi x}{3}}{6} = \frac{\Sigma y \cdot \sin \pi/3}{3} \quad \text{—— (4)}$$

x	y	$\cos\frac{\pi x}{3}$	$y \cdot \cos\left(\frac{\pi x}{3}\right)$	$\sin\left(\frac{\pi x}{3}\right)$	$y \cdot \sin\left(\frac{\pi x}{3}\right)$
0	9	1	9	0	0
1	18	0.5	9	0.866	15.58
2	24	-0.5	-12	+0.866	20.784
3	28	-1	-28	0	0
4	26	~~0.866~~ -0.5	-13	-0.866	-22.516
5	20	0.5	10	-0.866	-17.3
	$\Sigma y = 125$	Σ	$\Sigma y \cdot \cos\frac{\pi x}{3} = -25$		$\Sigma y \cdot \sin\frac{\pi x}{3} = -3.452$

$$a_0 = \frac{125}{3} = 41.66 \quad \text{—— (5)}$$

$$a_1 = \frac{-25}{3} = -8.33 \quad \text{—— (6)}$$

$$b_1 = \frac{\cancel{-3.42}\; -3.452}{3} = -1.15 \quad \text{—— (7)}$$

from (5), (6) and (7),

$$f(x) = \frac{41.66}{2} + (-8.33)\cos\frac{\pi x}{3} - 1.15 \sin\frac{\pi x}{3}$$

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Q1. $f(x) = x^2 + 2x + 1$ in $[0,1]$

① ~~x^2 is exponential~~, $2x$ is algebraic and ~~1 ans is~~
x^2+2x+1 is polynomial, therefore is continuous in $[0,1]$.

② $f(x) = x^2+2x+1$ in $[0,1]$
$f'(x) = 2x+2$
$\therefore$ $f(x)$ is differentiable in $(0,1)$.

$f(x)$ satisfies all conditions.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2c+2 = \frac{4-1}{1-0}$$

~~$2c+2$~~

$2(c+1) = 3$

$c = \frac{3}{2} - 1 = \frac{1}{2} = 0.5$

$c \in (0,1)$.

Q. $f(x) = 3 + 4x + 5x^2 + 7x^3$

$x - a = x - 1$

$a = 1$

$f(x) = f(a) + x f'(a) + \frac{x^2 f''(a)}{2!} + \frac{x^3 f'''(a)}{3!} + \ldots$

$f(a) = 3 + 4x + 5x^2 + 7x^3 \quad = 3+4+5+7 = 19$

$f'(a) = 4 + 10x + 21x^2 \quad = 4+10+21 = 35$

$f''(a) = 10 + 42x \quad = 10+42 \quad = 52$

$f'''(a) = 42 \quad = 42 \quad = 42$

$f(x) = 19 + (x-1)35 + \frac{(x-1)^2 52}{2} + \frac{(x-1)^3 42}{6}$

$= 19 + (x-1)35 + (x-1)^2 26 + (x-1)^3 7$

Q. $f(x+a) = 4 + (x+2) + 3(x+2)^2 - (x+2)^3$

Ans. $x + a = x + 2$

$a = 2$

$f(x) = f(a) + x f'(a) + \frac{x^2 f''(a)}{2!} + \frac{x^3 f'''(a)}{3!} + \ldots$

$f(a) = 4 + a + 3a^2 - a^3 \quad = 4+2+12-8 = 10$

$f'(a) = 1 + 6a - 3a^2 \quad = 1+12-12 = 1$

$f''(a) = 6 - 6a \quad = 6 - 12 \quad = -6$

$f'''(a) = -6 \quad = -6 \quad = -6$

$f^{iv}(a) = 0 \quad = 0 \quad = 0$

$f(x) = 10 + x + \frac{x^2(-6)}{2} + \frac{x^3(-6)}{6} =$

$f(x) = 10 + x - 3x^2 - x^3$

e^0 L

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Q. $\lim_{x \to \pi/2} \cos x^{\cos x}$

Ans. $\lim_{x \to \pi/2} \cancel{\cos 90}\ 0^0$

$\lim_{x \to \pi/2} \cos . x^{\cos x}\, dx = (\cos x^0 \cdot \cos x^{\cos x - 1})(-\sin x)$

NOTE:- if they are exponential, use log on both sides

$L = \lim_{x \to \pi/2} \cos x^{\cos x}$

$\log L = \lim_{x \to \pi/2} \log \cos x^{\cos x}$

$= \lim_{x \to 2} \cos x \cdot \log \cos x.$

$= \lim_{x \to 2} \dfrac{\log \cos x}{\sec x}$

L' Hospital Rule

$\log \cdot L = \lim_{x \to \pi/2} \boxed{\dfrac{\frac{-\sin x}{\cos x}}{\sec x \cdot \tan x}}$

$= \lim_{x \to \pi/2} \boxed{\dfrac{\cancel{-\tan x}}{\sec x \cdot \cancel{\tan x}}}$

$= \lim_{x \to \pi/2} -\cos x$

$\log L = 0$

$L = e^0$

$L = 1$

Q. $\dfrac{2}{1!} + \dfrac{3^2}{2!} + \dfrac{4^3}{3!} + \dfrac{5^4}{4!} + \dots$

OR

$\dfrac{2}{1!} + \dfrac{9}{2!} + \dfrac{64}{3!} + \dfrac{125}{4!} + \dots$

Let $u_n = \dfrac{(n+1)^n}{n!}$

$u_{n+1} = \dfrac{(n+2)^{n+1}}{(n+1)!}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{n!} \cdot \frac{(n+1)!}{(n+2)^{n+1}}$$

$$= \frac{(n+1)^n (n+1)}{(n+2)^{n+1}}$$

$$= \frac{(n+1)^n (n+1)}{(n+2)^n (n+2)}$$

$$= \left(\frac{n+1}{n+2}\right)^n \left(\frac{n+1}{n+2}\right)$$

$$= \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n$$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}$$

Q. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Ans. $u = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= (-x)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (-1)(x^2+y^2+z^2)^{-3/2}$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- } (1)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{3/2} \quad \text{--- } (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- } (3)$$

Adding (1), (2), and (3),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} =$$

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$= (x^2+y^2+z^2)^{-5/2}(3x^2+3y^2+3z^2) - (x^2+y^2+z^2)^{-3/2}(3)$

$= 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$

$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$

$= 0$

Q. If $u = \log(x^2+y^2)$, then verify that $u_{xy} = u_{yx}$

Ans. $u = \log(x^2+y^2)$

$u_{x\cancel{y}} = \dfrac{\partial u}{\partial x}$

$= \dfrac{\partial}{\partial x}\left[\log(x^2+y^2)\right]$

$= \dfrac{1}{x^2+y^2}(2x)$

$= \dfrac{2x}{x^2+y^2}$

$u_{yx} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$= \dfrac{\partial^2 u}{\partial y \cdot \partial x}$

$= \dfrac{\partial}{\partial y}\left(\dfrac{2x}{x^2+y^2}\right)$

$\dfrac{vu' - uv'}{v^2}$

$= 2x \dfrac{\partial}{\partial y}\left(\dfrac{1}{x^2+y^2}\right)$

$= 2x\left[\dfrac{-2x}{(x^2+y^2)^2}\right]$

$$u_y = \frac{\partial u}{\partial y}$$

$$= \frac{\partial}{\partial y}\left(\log(x^2+y^2)\right)$$

$$= \frac{1}{x^2+y^2}(2y)$$

$$= \frac{2y}{x^2+y^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)$$

$$= 2y \frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= 2y\left[\frac{-2x}{(x^2+y^2)^2}\right]$$

$$u_{xy} = \frac{-4xy}{(x^2+y^2)^2}$$

Q. $u = \frac{x^2+y^2}{x+y}$, then S.T. $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$

Ans. $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right).$$

$$= \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial (x+y)}{\partial x}}{(x+y)^2}$$

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$= \cancel{(x+y)\,2}$

$$= \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= 2x(x+y) - x^2 + y^2$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^2 + y^2 + 2xy}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{y^2 - x^2 + 2xy}{(x+y)^2}$$

Consider $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$

$$= \frac{2x^2 - 2y^2}{(x+y)^2} = \frac{2(x^2-y^2)}{(x+y)^2}$$

$$= \frac{2(x+y)(x-y)}{(x+y)^2}$$

$$= \frac{2(x-y)}{x+y}$$

$$\therefore \text{LHL} = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2$$

Consider $1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$

$$= 1 - \left(\frac{x^2 - y^2 + 2xy}{(x+y)^2}\right) - \left(\frac{y^2 + 2xy - x^2}{(x+y)^2}\right)$$

$$= 1 - x^2 + y^2 - 2xy - y^2 + 2xy + x^2$$

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Q. $u = \log(x^3 + y^3 + z^3 - 3xyz)$

Ans. $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$$L.H.S. = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) W$$

$$= \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z}$$

$$W = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

$$= \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3y^2 - 3xz)$$

$$= \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3z^2 - 3xy)$$

$$= \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xy}$$

$$W = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{(x+y+z)(x^2 + y^2 + z^2 - xy - xz - yz)}$$

$$W = \frac{3}{x+y+z}$$

$$\frac{\partial W}{\partial x} = \frac{-3(1)}{(x+y+z)^2}$$

$$\frac{\partial W}{\partial y} = \frac{-3}{(x+y+z)^2}$$

$$\frac{\partial W}{\partial z} = \frac{-3}{(x+y+z)^2}$$

from (1),

$$L.H.S. = \frac{-9}{(x+y+z)^2}$$

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Q. $u = \log(x^3 + y^3 - x^2 y - x y^2)$

Ans. $u = \log(x^3 - x^2 y + y^3 - x y^2)$

$= \log[x^2(x-y) + y^2(y-x)]$

$= \log[x^2(x-y) - y^2(x-y)]$

$= \log[(x-y)(x-y)(x+y)]$

$= \log[(x+y)(x-y)^2]$

$= \log(x+y) + \log(x-y)^2$

$= \log(x+y) + 2\log(x-y)$

$u(xt, yt) = t^1 u(x, y)$

$$u = \frac{x^2+y^2}{x+y}$$

$$= \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \sin^{-1}\left(\frac{x^2t^2+y^2t^2}{xt+yt}\right)$$

$$= \sin^{-1}\left[\frac{(x^2+y^2)t^2}{(x+y)t}\right]$$

$$\neq t \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

u is not homo.

$$\sin u = \frac{x^2+y^2}{x+y}$$

but $\sin u$ is homo fun. of degree 1.

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

Deduction from Euler's theorem

If $f(u)$ is a homo. func^n of degree n, in x a

then,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-$$

where $g(u) = \dfrac{nf(u)}{f'(u)}$

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Q. If $u = \log(x^3 + y^3 - x^2 y - x y^2)$ then find the value of

i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Ans. $u = \log(x^3 + y^3 - x^2 y - x y^2)$

put $x = xt$ and $y = yt$,

$u = \log [t^3 (x^3 + y^3 - x^2 y - x y^2)]$
$\neq u(x, y)$

$\therefore u$ is not homo. func.

But $f(u) = e^u =$
$= t^3 (x^3 + y^3 - x^2 y - x y^2)$

$\therefore f(u)$ is homogeneous function of degree 3.

By Euler's theorem,

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$

then $n = 3$.

$f(u) = e^u$
$f'(u) = e^u$

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$

$= 3 \frac{e^u}{e^u}$

$= 3$

and $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$

where $g(u) = \frac{n f(u)}{f'(u)} = 3$

$g'(u) = 0$

$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 3(0-1) = -3.$

Q. If $u = \sin^{-1}\sqrt{x^2+y^2}$, then show that.

$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \tan^3 u$

Ans. Put $x = xt$

$y = yt$

$u = \sin^{-1}(t\sqrt{x^2+y^2})$

$\neq u(x,y)$

$\therefore u$ is not homogeneous function.

But $f(u) = \sin u$

$= t\sqrt{x^2+y^2}$

$\therefore f(u)$ is a homo funcⁿ of degree 1.

By Euler's theorem,

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$

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$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u$$

and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where $g(u) = \frac{nf(u)}{f'(u)} = \tan u, \quad g'(u) = \sec^2 u$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}[13+\tan^3 u]$$

If $u = \csc^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$, then show that

Ans. $u = \csc^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Put $x = xt$ and $y = yt$.

$$u = \csc^{-1}\left[t^{1/12}\left(\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)\right] \neq u(x,y)$$

$\therefore u$ is not a homogeneous funcⁿ.

But $f(u) = \csc u = t^{1/12}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

is homogeneous function of degree $1/12$

$\therefore f(u)$ is homogeneous

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By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

then $n = \frac{1}{12}$

$f(u) = \csc u$
$f'(u) = -\csc u \cot u$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{12}\frac{\csc u}{-\csc u \cot u} = \frac{-1}{12\cot u}$$

$$= -\frac{\tan u}{12}$$

and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \frac{n f(u)}{f'(u)} = -\frac{1}{12}\tan u$

$g'(u) = \frac{-1}{12}\sec^2 u$

Q. If $u = \sin^{-1}\left[\sqrt{\frac{x^2+y^2}{x+y}}\right]$, then show that

$u = \sin^{-1}\sqrt{\frac{x^2+y^2}{x+y}}$

put $x = xt$
$y = yt$

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$$u = \sin^{-1}\sqrt{\frac{x^2t^2 + y^2t^2}{xt + yt}}$$

$$= \sin^{-1}\sqrt{\frac{t^2(x^2 + y^2)}{t(x+y)}}$$

$$= \sin^{-1}\sqrt{\frac{t(x^2+y^2)}{x+y}} \neq t\sin^{-1}\sqrt{\frac{x^2+y^2}{x+y}}$$

$$u \neq u(x,y)$$

$$f(u) = \sin^{-1}u$$

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Q. If $u = \tan^{-1}\left[\dfrac{x^3 + y^3}{x + y}\right]$ then show that,

Ans: $u = \tan^{-1}\left[\dfrac{x^3 + y^3}{x + y}\right]$

put $x = xt$ and
$y = yt$

Q. If $u = \sec^{-1}\left[\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right]$ the show that,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{1}{4} \cot u \left[3 + \cot^2 u\right]$$

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Jacobian:-

If $u(x,y)$ and $v(x,y)$ are differential func^ns of 2 independent variables x and y.

The Jacobian of (u,v) w/r to x, y.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarly if (u,v,w) are the func^ns of 3 independent variables $x, y,$ and z, then (u,v,w) w/r to x,y,z

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties

1. Jacobian of (u,v) w/r to (u,v)

$$\frac{\partial(u,v)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial v}{\partial u} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

2. If (u,v) are func^ns of (x,y) and are function of (x,y) and

$$J = \frac{\partial(u,v)}{\partial(x,y)} \quad \text{and} \quad J' = \frac{\partial(x,y)}{\partial(u,v)}$$

then $JJ' = 1$

3) Jacobian are functional determinant.

Q. If $u = x^2 - y^2$,
$v = 2xy$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

A. $u = x^2 - y^2$

$v = 2xy$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

Q. If $\cancel{x} = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial(u,v)}{\partial(x,y)}$.

Ans. $x = uv$

$y = \frac{u}{v}$

If $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$ then $JJ' = 1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix}$$

$$J = -\frac{2u}{v}$$

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$J' = \frac{1}{J} = \frac{-V}{2u}$

$= \frac{1}{2}\frac{V}{u}$

$= \frac{-1}{2y}$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{-1}{2y}$

Q. If $x = u(1-v)$ and $y = uv$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

Ans. $x = u(1-v) = u - uv$

$y = uv$

$x + y = u - uv + uv = u$

$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

$JJ' = 1$

$J' = \frac{1}{J}$

$J = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = (1-v)(u) - (v)(-u)$

$= u(1-v) + uv$

$= u - uv + uv$

$= u$

$= x + y$

Q. If $x = v^2 + w^2$

$y = w^2 + u^2$

$z = u^2 + v^2$ then find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Ans.

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$J' = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$= 0(0 - 4wv) - 2v(0 - 4uw) + 2w(4uv - 0)$

$= 8uvw + 8uvw$

$= 16uvw$

$$J' = \frac{1}{16uvw}$$

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Q. If $u = x(1-y)$, $v = xy$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Q. If $x = uv$, $y = \frac{u+v}{u-v}$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

Q. If $u = \frac{y-x}{1+xy}$, $v = \tan^{-1}y - \tan^{-1}x$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Q. If $x = e^v \sec u$, $y = e^v \tan$. then find $\frac{\partial(u,v)}{\partial(}$

Functional Dependence and Independence

Let $u = f(x, y)$ and $v = f(x, y)$ are func^n of x and y.

1) If $J = \dfrac{\partial(u, v)}{\partial(x, y)} = 0$ then,

(u,v) are functionally dependent.

Otherwise it is functionally independent.

2) If the given functions are functionally dependent, then there is a relation between them.

Q. Examine the functional dependence $u = \sin^{-1} + \sin^{-1} y$, $v = x\sqrt{1-y^2} + y\sqrt{4-x^2}$. Find the relation between them if it exists.

Ans. $J = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} \quad \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} \left(\dfrac{1}{\sqrt{1-x^2}}\right) & \left(\dfrac{1}{\sqrt{1-y^2}}\right) \\ \left(\sqrt{1-y^2} + \dfrac{y(-2x)}{2\sqrt{1-x^2}}\right) & \left(\dfrac{-2xy}{2\sqrt{1-y^2}} + \sqrt{1-x^2}\right) \end{vmatrix}$$

Finding the determinant,

$$J = \frac{1}{\sqrt{1-x^2}}\left(\frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right) - \frac{1}{\sqrt{1-y^2}}\left(\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right)$$

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$$= \frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}$$

$J = 0$

$\because J = 0$

$\therefore (u, v)$ are functionally independent.

(Relation):

Let $\sin^{-1}x = \alpha$,

$\sin^{-1}y = \beta$

$\Rightarrow \quad x = \sin\alpha$

$y = \sin\beta$

$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$= \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}$

$= \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \sin(\alpha+\beta)$

$= \sin(\sin^{-1}x + \sin^{-1}y)$

$= \sin u$

H.W

Q. Examine the functional dependence $u = \sin^{-1}x - \sin^{-1}y$ and $v = x\sqrt{1-y^2} - y\sqrt{1-x^2}$. Find the relation between them if it exists.

Q. Examine for the functional dependence $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$. Find the relation between if it exists.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

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$$= \begin{vmatrix} \dfrac{(1-xy)(1)+(-y)(x+y)}{(1-xy)^2} & \dfrac{(1-xy)(1)-(-x)(x+y)}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{(1-xy)-y(x+y)}{(1-xy)^2} & \dfrac{(1-xy)+x(x+y)}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

(Relation):

Let $\tan^{-1}x = \alpha$, $\tan^{-1}y = \beta$.

$\Rightarrow x = \tan\alpha$

$y = \tan\beta$

$$u = \frac{x+y}{1-xy}$$

$$= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha \cdot \tan\beta}$$

$$= \tan(\alpha+\beta)$$

$$= \tan(\tan^{-1}x + \tan^{-1}y)$$

$$u = \tan v$$

Q. Examine for functional dependency of $u = \dfrac{x-y}{1+xy}$ and $v = \tan^{-1}x - \tan^{-1}y$. Find the relation between them exists.

Ans. $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

=	$\dfrac{(1)(1+xy)-(x-y)(y)}{(1+xy)^2}$	$\dfrac{(-1)(1+xy)-(x-y)(x)}{(1+xy)^2}$
	$\dfrac{1}{1+x^2}$	$-\dfrac{1}{1+y^2}$

$J =$

$$J = \frac{(1+xy)-y(x-y)}{(1+xy)^2}\cdot\frac{-1}{1+y^2} - \frac{(-1)(1+xy)-x(x-y)}{(1+xy)^2}\left(\frac{1}{1+x^2}\right)$$

Q. $u = x+y+z$

$V = x^2+y^2+z^2$

$W = xy+yz+zx$

Ans.

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$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y & z & x \end{vmatrix}$$

$$J = 1(2xy - 2z^2) - 1(2x^2 - 2yz) + 1(2xz - 2y^2)$$
$$= 2xy - 2z^2 - 2x^2 + 2yz + 2xz - 2y^2$$
$$= -2x^2 - 2y^2 - 2z^2 + 2xy + 2yz + 2xz$$
$$= -2\,(x^2 + y^2 + z^2 - xy - yz - xz)$$

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Errors and Approximations

Let $u = f(x,y)$ be a continuous and differentiable func^n of 2 independent variables x and y and the total derivative of u is denoted by du.

$$du = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$$

If ∂x, ∂y and ∂u are errors in x, y, u respectively, then

1) dx, dy, du are known as absolute error in x, y, u respectively.

2) $\frac{dx}{x}$, $\frac{dy}{y}$, $\frac{du}{u}$ are known as relative error.

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in x, y, z respectively

3) $\frac{dx}{x} \times 100$, $\frac{dy}{y} \times 100$, $\frac{dz}{z} \times 100$ are known as % error in x, y, z respectively.

4) $x + dx$, $y + dy$, $z + dz$ are known as approximate value of x, y, z respectively.

Q. Find the % error in computing the parallel ~~resistors~~ ~~resistors R~~ resistance R of two Resistors R_1 and R_2 from the following formula.

$\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2}$ where R_1 and R_2 are both in error by 3% each

Ans. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

Differentiating both sides,

$$-\frac{1}{r^2} dr = -\frac{1}{r_1^2} dr_1 - \frac{1}{r_2^2} dr_2$$

$$-\frac{1}{r} \frac{dr}{r} = -\frac{1}{r_1} \cdot \frac{dr_1}{r_1} - \frac{1}{r_2} \cdot \frac{dr_2}{r_2}$$

$$-\frac{1}{r}\left(\frac{dr}{r}\right) = -\left[\frac{1}{r_1}\left(\frac{dr_1}{r_1}\right) + \frac{1}{r_2}\left(\frac{dr_2}{r_2}\right)\right]$$

Multiplying by 100,

$$\frac{1}{r}\left(\frac{dr}{r} \times 100\right) = \frac{1}{r_1}\left(\frac{dr_1}{r_1} \times 100\right) + \frac{1}{r_2}\left(\frac{dr_2}{r_2} \times 100\right)$$

$$\frac{1}{r}\left(\frac{dr \times 100}{r}\right) = \frac{3}{r_1} + \frac{3}{r_2}$$

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$$\frac{1}{r}\left(\frac{dr}{r}\times 100\right) = 3\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

But $\frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}$

$$\frac{1}{r}\left(\frac{dr}{r}\times 100\right) = 3\left(\frac{1}{r}\right)$$

$$\therefore \frac{dr}{r}\times 100 = 3$$

$\therefore$ % error = 3%

Q. If focal length of mirror is found from the following formula.

$$\frac{1}{v}-\frac{1}{u} = \frac{2}{f}$$

Find % error in f if u and v are both in error by 2% each

$$-\frac{1}{v^2}dv + \frac{1}{u^2}du = \frac{-2}{f^2}df$$

$$-\left(\frac{1}{v}\cdot\frac{dv}{v} - \frac{1}{u}\cdot\frac{du}{u}\right) = \frac{-2}{f}\cdot\frac{df}{f}$$

$$\frac{1}{v}\cdot\frac{dv}{v}\times 100 - \frac{1}{u}\cdot\frac{du}{u}\times 100 = \frac{+2}{f}\cdot\frac{df}{f}\times 100$$

$$\frac{2}{v} - \frac{2}{u} = +\frac{2}{f}\left(\frac{df}{f}\times 100\right)$$ [Given]

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$$\frac{1}{v} - \frac{1}{u} = \pm \frac{1}{f}\left(\frac{df}{f}\times 100\right)$$

$$\frac{2}{f} = \pm \frac{1}{f}\left(\frac{df}{f}\times 100\right)$$

$$\therefore \left(\frac{df}{f}\times 100\right) = 2$$

$\therefore$ % error = 2%.

Q. Find the % error in the area of an ellipse when an error of 2% is made in ~~meaus~~ measuring in major axis and 1% made in measuring its minor axis.

Ans. ~~ar (ellipse) = $\pi(R_1+R_2)^2$~~

Let A be area and ~~2A~~ 2a and ~~2B~~ be the major and minor axis having eq^n

$$A = \pi \cdot a \cdot b$$

Taking log on both sides,

$$\log A = \cancel{\pi} \log(a \cdot b\pi)$$

$$\log A = \log \pi + \log a + \log b$$

$$\frac{1}{A} dA = 0 + \frac{da}{a} + \frac{db}{b}$$

$$\frac{dA}{A}\times 100 = \frac{da}{\cancel{da}}\times 100 + \frac{db}{b}\times 100$$

$$\frac{dA}{A}\times 100 = 2 + 1$$

$$\frac{dA}{A}\times 100 = 3$$

$\boxed{\therefore \text{ \% error is } 3\%.}$

Q. In calculating the volume of right circular cone, error of 2% and 1% are made in measuring the height and radius. Find the % error in the calculated volume.

Ans. Let V be the volume of cone.

$\Rightarrow V = \frac{\pi r^2 h}{3}$

Taking log on both sides

$\log V = \log\left(\frac{\pi r^2 h}{3}\right)$

$\log V = \log \pi + \log r^2 + \log h + \log \frac{1}{3}$

$\log V = \log \pi + 2\log r + \log h + \log \frac{1}{3}$

$\log V = 0 + 2\log r + \log h + 0$

Differentiating,

$\frac{1}{V}dV = \frac{2}{r}dr + \frac{1}{h}dh$

$\frac{dV}{V} \times 100 = 2\left(\frac{dr}{r} \times 100\right) + \left(\frac{dh}{h} \times 100\right)$

$\frac{dV}{V} \times 100 = 2(1) + 2$

$\frac{dV}{V} \times 100 = 4\%$

$\therefore$ % error is 4%.

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Maxima and Minima (func^n of 2 independent variables)

Let $f(x,y)$ be a func^n of 2 independent variables x and y and it is continuous and differentiable. To find max or min value, use following procedure

1. Given func^n $f(x,y)$

2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

3. Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

4. Solve these eq^ns simultaneously and find the value of x and y.

5. $(x,y) \equiv (a_1, b_1), (a_2, b_2), \ldots$ are called stationary points.

5. Find $r = \frac{\partial^2 f}{\partial x^2}$

$s = \frac{\partial^2 f}{\partial x \partial y}$

$t = \frac{\partial^2 f}{\partial y^2}$

6. Find r, s, t for each pair.

For (a_1, b_1),

Find (a_1, b_1), $s(a_1, b_1)$, and $t(a_1, b_1)$

7. $rt - s^2$ in (a_1, b_1)

$rt - s^2$

$rt - s^2 < 0$
- → not min
- → not max
- → (a_1, b_1) = saddle point

$rt - s^2 = 0$
- → more investigation needed

$rt - s^2 > 0$
- → $r > 0$
  - → func^n is min at (a_1, b_1)
- → $r < 0$
  - → func^n is max at (a_1, b_1)

- Exteme points. .
  - → $F_{min}(a_1, b_1)$
  - → $F_{max}(a_1, b_1)$

Q. Find the extreme values of $f(x,y) = x^2 + y^2 + 6x + 24$

Ans. $f(x,y) = x^2 + y^2 + 6x + 24$

For stationary point,

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 2x + 6 = 0$$

$$\therefore x = -3$$

$$\frac{\partial f}{\partial y} = 2y = 0$$

$$\therefore y = 0$$

$\therefore$ The stationary point $(x, y) \equiv (-3, 0)$.

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x+6) := 2$$

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$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2y) = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(2y) = 2$$

$r(-3,0)$

$\therefore r(-3,0) = 2$

$s(-3,0) = 0$

$t(-3,0) = 2$

$$rt - s^2 = (2)(2) - (0)^2$$

$$= 4 > 0$$

$r = 2 > 0$.

$\therefore$ given funcⁿ is minimum at $(-3,0)$.

From min value,

$$f_{min} = f(-3,0)$$

$$= (-3)^2 + (0)^2 + 6(-3) + 24$$

$$= 9 - 18 + 24 \qquad 33 - 18$$

$$= 15 \qquad 33$$

$$\boxed{\therefore f_{min} = 15}$$

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Q. Find the extreme value of $f(x,y) = x^2 + 2y^2 + 4y - 21$

Ans.

For stationary point

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$

$$\frac{\partial f}{\partial x} = 2x = 0$$
$$x = 0$$

$$\frac{\partial f}{\partial y} = -4y + 4 = 0$$
$$y = -1$$

$\therefore$ The stationary point $(x, y) = (0, -1)$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left(+4y + 4\right)$$
$$= +4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$= 0$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}(2x) = 2$$

$$rt - s^2 = (2)(+4) - (0)^2$$
$$= +8 < 0$$

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$\therefore$ func^n is max at $(0,-1)$
$r = 2 > 0$

For min value,
$f_{min} = f(0,-1)$
$= (0)^2 + 2(-1)^2 + 4(-1) - 21$
$= +2 - 4 - 21$
$= -23$

$f(x,y) = 10 - 4x - 2x^2 - 4y^2$
For stationary point $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

~~$2x = 0$ and 2~~
$\frac{\partial f}{\partial x} = 0 = -4 - 4x$
$x = -1$

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$\frac{\partial f}{\partial y} = 0 = -8y$
$y = 0$

For stationary point $\equiv (-1,0)$

$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(-4-4x)$
$= -4$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-8y) = 0$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-8y) = -8$

$rt - s^2 = (-4)(-8) - (0)^2$
$= 32 > 0$

$\therefore f_{min} = f(-1, 0)$
$= 10 - 4$

Unit I:

1) Partial der.: type I
2) Variable treated as constant
3) Euler's theorem u and f(u)
4) composite func. substitution
5) λ and μ
6) linear dep/indep
7) Matrix rank

SharkCoders

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Unit II:

1) Eigen values
2) Diagonalisation

Keep Smile

Unit IV:

1) Harmonic analysis
2) Ratio Test
3) Half fourier series